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SCIENCE DIRECT
Journal of Sound and Vibration 275 (2004) 769–794
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Statistical model-based damage detection and localization: subspace-based residuals and damage-to-noise sensitivity ratios

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Received 29 November 2002; accepted 1 July 2003

Abstract

The vibration-based structural health monitoring problem is addressed as the double task of detecting damages modelled as changes in the eigenstructure of a linear dynamic system, and localizing the detected damages within (a finite element model (FEM) of) the monitored structure. The proposed damage detection algorithm is based on a residual generated from a stochastic subspace-based covariance-driven identification method and on the statistical local approach to the design of detection algorithms. This algorithm basically computes a global test, which performs a sensitivity analysis of the residuals to the damages, relative to uncertainties and noises. How this residual relates to some residuals for damage localization and model updating is discussed.

Damage localization is stated as a detection problem. This problem is addressed by plugging aggregated sensitivities of the modes and mode shapes with respect to FEM structural parameters in the above setting. This results in directional tests, which perform the same type of damage-to-noise sensitivity analysis of the residual as for damage detection. How the sensitivity aggregation mechanism relates to sub-structuring is outlined.

1. Introduction

Detecting and localizing damages for monitoring the integrity of structural and mechanical systems is a topic of growing interest, due to the aging of many engineering structures and machines and to increased safety norms. Automatic global vibration-based monitoring techniques

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.07.016

turn out to be useful alternatives to visual inspections or local non-destructive (e.g., ultrasonic) evaluations performed manually.

Health monitoring techniques based on processing vibration measurements basically handle two types of characteristics: the *structural parameters* (mass, stiffness, flexibility, damping) and the *modal parameters* (modal frequencies, and associated damping values and mode shapes); see Refs. [1–3] and references therein. A central question for monitoring is to compute *changes* in those characteristics and to assess their *significance*. For the *frequencies*, the crucial issues are then: how to compute the changes, to assess that the changes are significant, and to handle *correlations* among individual changes. A related issue is how to compare the changes in the frequencies obtained from experimental data with the sensitivity of modal parameters obtained from an analytical model. Furthermore, it has been widely acknowledged that, whereas changes in frequencies bear useful information for damage *detection*, information on changes in (the curvature of) mode shapes is mandatory for performing damage *localization*. Then, similar issues arise for the computation and the significance of the changes. In particular, assessing the significance of (usually small) changes in the mode shapes, and handling the (usually high) correlations among individual mode shape changes are still considered as open questions [1–4].

Controlling the computational complexity of the processing of the collected data is another standard monitoring requirement, which includes limited use of an analytical model of the structure. Moreover, the reduction from the analytical model to the experimental model (truncated modal space) is known to play a key role in the success of model-based damage detection and localization [1,5].

The purpose of the present paper is to describe the foundations and analyze the properties of a damage detection and localization method. This method is based on an approach which aims at addressing the issues and overcoming (some of) the limitations above. It is assumed that a signature of the structure in its nominal (safe) state is available, typically a moderate number of modes and a moderate number of components of associated mode shapes. This signature is usually identified using reference data, possibly recorded under an unknown non-stationary excitation. The proposed algorithm processes new data by first generating a residual, and computing its sensitivity with respect to damages. The residual is shown to be asymptotically Gaussian under both no damage and small damage assumptions. This results in a global test, which performs a *sensitivity* analysis of the residual to the damages, *relative to* uncertainties in the modal estimates and noises on the available data. Modal diagnosis is stated as a detection problem: deciding which components of the modal parameter vector θ have changed. This problem is solved by designing similar χ^2-tests focussed onto the modal subspaces of interest. Damage localization is stated as a detection problem, and not an (usually ill-posed) inverse estimation problem. This problem is addressed by plugging aggregated sensitivities of the modes and mode shapes with respect to finite element model (FEM) structural parameters in the above setting. This results in directional tests, which perform the same type of damage-to-noise sensitivity analysis of the residual as for damage detection. The computation, the analytical-to-experimental matching and the aggregation of the sensitivities are performed off-line at a design stage, whereas the directional tests may be computed on-board. Several key concepts and techniques of this approach have already been published by the same group of authors [6–12]. However no journal paper containing the details of the damage localization method has been published. Moreover, it is one of the purposes of this paper to address crucial issues and well-known obstacles of model-based vibration monitoring.

The paper is organized as follows. In Section 2, the modelling issues are introduced and some key parameterizations and sensor type issues are discussed. Section 3 is devoted to the proposed damage detection method, based on a general statistical local approach to the detection of small deviations in the parameters of dynamic systems, and on a stochastic subspace-based covariance-driven modal identification method. The off-line computation and the on-board analysis stage are distinguished, and the effect of a truncated modal space is briefly discussed. In Section 4, the modal diagnosis method, which handles residual sensitivities to damages and residuals uncertainties is described. Damage localization is addressed in Section 5, where the off-line computation and the on-board analysis stages are distinguished again. How the proposed method relates to other works is addressed in Section 6, where residual design and structural aggregation are discussed. Practical constraints and lessons learnt from application examples are also commented on in this section. Some conclusions are drawn in Section 7.

2. Modelling and parameterizations

First, the main equations and parameters of the models which are used are recalled. Identifiable and non-identifiable models are distinguished, and a useful invariance property of the modal parameters is outlined. Then the effect of changing the sensor types is discussed. Finally, the damage detection and localization problems investigated throughout are stated.

2.1. Dynamical model and structural parameters

It is assumed that the behaviour of the mechanical system can be described by a stationary linear dynamical system, and that, in the frequency range of interest, the input forces can be modelled as a non-stationary white noise. This results in:

$$M\ddot{\mathscr{Z}}(t) + C\dot{\mathscr{Z}}(t) + K\mathscr{Z}(t) = v(t), \quad Y(t) = L\mathscr{Z}(t), \tag{1}$$

where t denotes continuous time, M, C, K are the mass, damping and stiffness matrices, respectively, (high-dimensional) vector $\mathscr{Z}$ collects the displacements of the degrees-of-freedom of the structure; the external (non-measured) force v is modelled as a non-stationary white noise with time-varying covariance matrix $Q_v(t)$, measurements are collected in the (often, low dimensional) vector Y, and matrix L indicates which components of the state vector are actually measured (where the sensors are located).

The modes or eigenfrequencies denoted generically by μ, the eigenvectors ϕ_μ, and the mode shapes denoted generically by ψ_μ, are solutions of

$$\det(\mu^2 M + \mu C + K) = 0, \quad (\mu^2 M + \mu C + K)\phi_\mu = 0, \quad \psi_\mu = L\phi_\mu. \tag{2}$$

The frequency and damping coefficient are recovered from a *continuous* eigenvalue μ through

$$\text{Frequency } f = \frac{b}{2\pi}, \text{ Damping } d = -\frac{a}{\sqrt{a^2 + b^2}}, \text{ where } a = \mathrm{Re}(\mu), b = \mathrm{Im}(\mu) \tag{3}$$

Some comments are in order on parameterizations of interest for damage detection and localization. Since a local damage in the structure reduces the stiffness and increases the damping, many damage detection techniques have been proposed which monitor the stiffness matrix K. Monitoring its inverse K^{-1}, namely the flexibility matrix, has proven more tractable and computationally feasible [1,4,5]. In some cases, other structural parameterizations such as volumic mass and Young elasticity modulus may be preferable [13,1]. Also, several methods in the literature are based on a transmissibility matrix [3,14], which involves the processing of input–output data. However, in the case of non-measured input excitation, processing output-only data is mandatory [15,16]. On the other hand, a reduced stiffness and an increased damping result in decreased natural frequencies and modified mode-shapes geometry. Thus, monitoring the modal parameters is relevant, at least for damage detection, especially since the modal parameters can be estimated by processing output-only data [16]. Damage localization, however, requires (at least partial) knowledge of structural parameters and geometry.

Consequently, the proposed damage detection method handles the modal parameters, which enjoy a useful invariance property, as explained next. Moreover, the proposed method does *not* make use of any modal parameters extracted from an analytical model, but uses identified modal parameters instead. The proposed damage localization method handles both modal and structural parameters, using an original structural aggregation mechanism.

2.2. State-space model and canonical parameterization

Sampling model (1) at rate $1/\tau$ yields the discrete time model in state-space form [17,18]:

$$\begin{aligned} X_{k+1} &= FX_k + V_{k+1}, \\ Y_k &= HX_k, \end{aligned} \tag{4}$$

where the state and the output are

$$X_k = \begin{pmatrix} \mathscr{Z}(k\tau) \\ \dot{\mathscr{Z}}(k\tau) \end{pmatrix}, \; Y_k = Y(k\tau), \tag{5}$$

the state transition and observation matrices are

$$F = \mathrm{e}^{\mathscr{L}\tau}, \mathscr{L} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$$

and

$$H = (L \quad 0). \tag{6}$$

In Eq. (4), the *unmeasured* state noise V_{k+1} is assumed to be Gaussian, zero-mean, white, with covariance matrix:

$$Q_{k+1} \stackrel{\text{def}}{=} \mathbf{E}(V_{k+1}\, V_{k+1}^{\mathrm{T}}) = \int_{k\tau}^{(k+1)\tau} \mathrm{e}^{\mathscr{L}s} \tilde{Q}(s) \mathrm{e}^{\mathscr{L}^{\mathrm{T}}s}\, \mathrm{d}s,$$

where $\mathbf{E}(.)$ denotes the expectation operator and

$$\tilde{Q}(s) = \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}Q_v(s)M^{-\mathrm{T}} \end{pmatrix}.$$

The whiteness assumption on the state noise and the absence of measurement noise in Eq. (4) are discussed in Ref. [19]. It is stressed that sinusoidal or coloured noise excitation can be encompassed as well. State X and observed output Y have dimensions $2m$ and r, respectively, with r (often much) smaller than $2m$ in practice.

Let (λ, ϕ_λ) be the eigenstructure of the state transition matrix F, namely

$$\det(F - \lambda I) = 0, \quad (F - \lambda I)\phi_\lambda = 0. \tag{7}$$

The modal parameters (μ, ψ_μ) in Eq. (2) can be deduced from the (λ, ϕ_λ)'s in Eq. (7) using

$$e^{\tau\mu} = \lambda, \quad \psi_\mu = \varphi_\lambda \stackrel{\text{def}}{=} H\phi_\lambda. \tag{8}$$

The frequency and damping coefficient are recovered from a discrete eigenvalue λ through

$$\begin{aligned} &\text{Frequency} = \frac{a}{2\pi\tau}, \ \text{Damping} = \frac{|b|}{\sqrt{a^2 + b^2}}, \\ &\text{where } a = \left|\arctan\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right|, \ b = \ln|\lambda|. \end{aligned} \tag{9}$$

Eigenvectors are real if $C = \alpha M + \beta K$, the simplest form of proportional damping. Because of the structure of the state in Eq. (5), the λ's and φ_λ's are pairwise complex conjugate.

It turns out that the collection of modes $(\lambda, \varphi_\lambda)$, which form a very natural parameterization for structural analysis, also enjoy a nice invariance property. Actually, it can easily be shown to be invariant with respect to changes in the state basis of system (4). In other words, the $(\lambda, \varphi_\lambda)$'s form a canonical parameterization of the eigenstructure (or equivalently the pole part) of that system. Let the $(\lambda, \varphi_\lambda)$'s be stacked into a $(r+1)m$-dimensional vector θ:

$$\theta \stackrel{\text{def}}{=} \begin{pmatrix} \Lambda \\ vec\ \Phi \end{pmatrix}, \tag{10}$$

where Λ is the vector whose elements are the eigenvalues λ, Φ is the matrix whose columns are the mode shapes φ_λ's, and *vec* is the column stacking operator. From now on, vector θ is considered as the system parameter.

It should be stressed that it is not needed to favour a particular normalization of the mode shapes ϕ_μ's and thus of the φ_λ's, as opposite to the mass normalized modes: $\phi_\mu^{\mathrm{T}} M \phi_\mu = I$ used in e.g. Ref. [5]. As explained below, the proposed damage-to-noise sensitivity ratios are invariant with respect to such normalizations. However, some care should be taken for damage localization, while matching identified mode shapes with analytical ones as discussed in 5.2.3.

2.3. Different sensor types

The measurement equation in Eq. (4) with H as in Eq. (6) implicitly assumes that the available sensors measure the (relative) displacements of the degrees-of-freedom themselves, namely are constraint gauges. If constraint gauges, velocity sensors and accelerometers are available, the measurement equation in Eq. (1) should write

$$Y(t) = \begin{pmatrix} L\mathscr{Z}(t) \\ N\dot{\mathscr{Z}}(t) \\ P\ddot{\mathscr{Z}}(t) \end{pmatrix}$$

with L, N, P made of 0's and 1's, and system (5) should be understood with

$$H = \begin{pmatrix} L & 0 \\ 0 & N \\ -PM^{-1}K & -PM^{-1}C \end{pmatrix}.$$

Consequently, state-space model (4) can always be enforced, whatever the sensors are. The nature of the sensors used only influences the observation matrix H.

2.4. The damage detection and localization problems

In this paper, damage detection is stated as the problem of detecting changes in the canonical parameter vector θ defined in Eq. (10). It is assumed that a reference value θ_0 is available. Generally, such a reference parameter is identified using data recorded on the undamaged system. Of course, when the monitored system is subject to non-stationary input excitation, the reference value θ_0 should be identified on long data samples containing as many of these nuisance changes as possible. However, it is important to note that, with the proposed method, the detection problem may be solved on the basis of data samples of much smaller size.

Given, on one hand, a reference value θ_0 of the model parameter and, on the other hand, a new data sample, the detection problem is to decide whether the new data are still well described by this parameter value or not. The modal diagnosis problem is to decide which components of the modal parameter vector θ have changed. The damage localization problem is to decide which parts of the structure have been damaged, or equivalently to decide which elements of the structural parameter matrices have changed.

Because structural identification is a complex (and generally not fully automatic) process, and because it is intended to design a damage detection algorithm which can be run on-board, our approach does *not* perform a new parameter estimation using the new data sample. Instead, the damage detection and modal diagnosis problems are solved through the on-board computation and analysis of a residual. Similarly, because structural model updating is a computationally involved procedure, the damage localization problem is not addressed as an (usually ill-posed inverse) estimation problem, but as an evaluation of the correlations of this residual with specific structural parameter subspaces. This is explained in the next three sections.

3. Damage detection

The design of the proposed damage detection algorithm is based on a general statistical approach, which aims at transforming a large class of detection problems concerning a parameterized *stochastic process* into the universal problem of monitoring the mean of a Gaussian *random vector* [7]. This approach basically addresses the early warning of *small deviations* of the system parameter. The key idea is to define a convenient residual, tightly associated with a relevant parameter estimation method, and to compute the sensitivity of the residual with respect to damages (viewed as changes in the parameter vector) and the uncertainty in the residual due to process noise and estimation errors. Moreover, the residual can be shown to be asymptotically Gaussian. Hence the analysis of the residual's sensitivity to the damages relative to uncertainties and noises is easy: a sound decision rule can be designed for assessing whether the residual has *significantly* deviated from zero or not.

For structural vibration monitoring and damage detection, the main issue is thus the definition of a parameter estimating function associated with a modal identification algorithm. The use of an output-only and covariance-driven subspace-based stochastic identification has been advocated [16,19]. The residual corresponding to this method is introduced in 3.1. The handling of the residual sensitivities and uncertainty is addressed in 3.2, and their off-line computation described in 3.3. The residual analysis and the resulting on-board damage detection algorithm are given in 3.4.

3.1. Residual associated with stochastic subspace identification

The key steps of the subspace structural identification method are briefly recalled. A characterization of the modal parameter in Eq. (10) is exhibited, from which the proposed residual can be defined. Finally, the effect of a truncated modal space is discussed.

3.1.1. Output-only covariance-driven subspace identification

Covariance-driven subspace identification of the eigenstructure $(\lambda, \varphi_\lambda)$'s is based on the following steps. Let $R_i \stackrel{\text{def}}{=} \mathbf{E}(Y_k\, Y_{k-i}^{\mathrm{T}})$ and

$$\mathscr{H}_{p+1,q} \stackrel{\text{def}}{=} \begin{pmatrix} R_0 & R_1 & \vdots & R_{q-1} \\ R_1 & R_2 & \vdots & R_q \\ \vdots & \vdots & \vdots & \vdots \\ R_p & R_{p+1} & \vdots & R_{p+q-1} \end{pmatrix} \stackrel{\text{def}}{=} Hank(R_i) \tag{11}$$

be the output covariance and Hankel matrices, respectively. Introducing the cross-covariance between the state and the observed outputs: $G \stackrel{\text{def}}{=} \mathbf{E}(X_k\, Y_k^{\mathrm{T}})$, direct computations of the R_i's from Eqs. (4) lead to: $R_i = HF^iG$, and to the well-known [20] factorization

$$\mathscr{H}_{p+1,q} = \mathscr{O}_{p+1}(H,F)\mathscr{C}_q(F,G), \tag{12}$$

where

$$\mathscr{O}_{p+1}(H,F) \stackrel{\text{def}}{=} \begin{pmatrix} H \\ HF \\ \vdots \\ HF^p \end{pmatrix} \quad \text{and} \quad \mathscr{C}_q(F,G) \stackrel{\text{def}}{=} (G\; FG \cdots F^{q-1}G) \tag{13}$$

are the observability and controllability matrices, respectively. The observation matrix H is then found in the first block-row of the observability matrix $\mathcal{O}$. The state-transition matrix F is obtained from the shift invariance property of $\mathcal{O}$, namely

$$\mathcal{O}_p^{\uparrow}(H,F) = \mathcal{O}_p(H,F)F, \text{ where } \mathcal{O}_p^{\uparrow}(H,F) \stackrel{\text{def}}{=} \begin{pmatrix} HF \\ HF^2 \\ \vdots \\ HF^p \end{pmatrix}.$$

Assuming $rank(\mathcal{O}_p) = \dim F$, and thus that the number of block-rows in $\mathcal{H}_{p+1,q}$ is large enough, is mandatory for recovering F. The eigenstructure (λ, ϕ_λ) then results from Eq. (7).

The actual implementation of this subspace algorithm, known under the name of balanced realization (BR) [21,22], processes the empirical covariance and Hankel matrices

$$\hat{R}_i \stackrel{\text{def}}{=} 1/n \sum_{k=1}^{n} Y_k Y_{k-i}^{\mathrm{T}}, \quad \hat{\mathcal{H}}_{p+1,q} \stackrel{\text{def}}{=} Hank(\hat{R}_i) \tag{14}$$

and exploits the well-known subspace interpretation of the singular value decomposition (SVD) [23]: the SVD of $\hat{\mathcal{H}}_{p+1,q}$—possibly pre- and post-weighted [24]—and its truncation at the desired model order yield, in the left factor, an estimate $\hat{\mathcal{O}}$ for the observability matrix $\mathcal{O}$:

$$W_1 \hat{\mathcal{H}} W_2^{\mathrm{T}} = UDV^{\mathrm{T}} = U \begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} V^{\mathrm{T}},$$

$$\hat{\mathcal{O}} = W_1^{-1}\, UD_1^{1/2}, \quad \hat{\mathcal{C}} = D_1^{1/2}\, V^{\mathrm{T}}\, W_2^{-\mathrm{T}},$$

where W_1 and W_2 are invertible weighting matrices (design parameters). From $\hat{\mathcal{O}}$, estimates $(\hat{H}, \hat{F})$ and $(\hat{\lambda}, \hat{\phi}_\lambda)$ are recovered as sketched above. How to select the number of lags $(p+q)$ and thus the size of $\hat{\mathcal{H}}_{p+1,q}$ is discussed in Refs. [16,19].

The key feature in this algorithm is factorization (12), where the left factor $\mathcal{O}$ only depends on the pair (H,F), and thus on the modal parameters $(\lambda, \varphi_\lambda)$.

3.1.2. *Exploiting a canonical parameter characterization*

Factorization (12) is the key for a characterization of the canonical parameter vector θ in Eq. (10), and for deriving the parameter estimating function implicitly used in the above subspace identification algorithm.

Assume that the eigenvectors of matrix F are chosen as a basis for the state space of model (4). In this basis, the observability matrix in Eq. (13) writes [8]

$$\mathcal{O}_{p+1}(\theta) = \begin{pmatrix} \Phi \\ \Phi\Delta \\ \vdots \\ \Phi\Delta^p \end{pmatrix}, \tag{15}$$

where diagonal matrix Δ is defined as $\Delta = \mathrm{diag}(\Lambda)$, and Λ and Φ are as in Eq. (10). Then the following property results from factorization (12). Whether a nominal parameter θ_0 is in agreement with a given output covariance sequence $(R_j)_j$ is characterized by [25,9]

$$\mathcal{O}_{p+1}(\theta_0) \text{ and } \mathcal{H}_{p+1,q} \text{ have the same left kernel space.} \tag{16}$$

The left kernel space of matrix M is the kernel space of matrix M^T. This property can be checked as follows. From the nominal modal parameter vector θ_0, compute $\mathcal{O}_{p+1}(\theta_0)$ using Eq. (15), and perform e.g. a SVD of $W_1\mathcal{O}_{p+1}(\theta_0)$ for defining its left kernel space, namely extracting an orthonormal matrix S such that $S^T S = I_s$ and

$$S^T W_1 \mathcal{O}_{p+1}(\theta_0) = 0. \tag{17}$$

Matrix S depends implicitly on parameter θ_0. It is not unique—two such matrices are related through a post-multiplication with an orthonormal matrix U. Nevertheless, for reasons which are made clear below, S can be treated as a function of θ_0, denoted by $S(\theta_0)$. Then the characteristic property (16) writes

$$U^T S^T(\theta_0) W_1 \mathcal{H}_{p+1,q} W_2^T = 0, \tag{18}$$

where W_1 and W_2 are invertible weighting matrices as before.

Assume now that a reference parameter θ_0 and a new data sample $Y_1, \ldots, Y_n$ are available. For checking whether the data are well described by θ_0, the idea is to compute the empirical covariance sequence and fill in the empirical block-Hankel matrix $\hat{\mathcal{H}}_{p+1,q}$ using Eq. (14), and to define the residual vector:[1]

$$\zeta_n(\theta_0) \stackrel{\text{def}}{=} \sqrt{n}\, vec(U^T S^T(\theta_0) W_1 \hat{\mathcal{H}}_{p+1,q} W_2^T). \tag{19}$$

Let θ be the actual value of the parameter for the system which generated the new data sample, and $\mathbf{E}_\theta$ be the expectation when the actual parameter is θ. It results from Eq. (18) that

$$\mathbf{E}_\theta(\zeta_n(\theta_0)) = 0 \quad \text{iff } \theta = \theta_0. \tag{20}$$

In other words, vector $\zeta_n(\theta_0)$ in Eq. (19) has zero mean in the absence of change in θ, and non-zero mean in the presence of a change (damage). Consequently it plays the role of a residual. The question is then how to decide that the residual ζ_n is *significantly* different from zero. In particular, the sensitivity of the residual with respect to deviations in the modal parameter should be compared with the fluctuations of the residual around its zero mean. This is discussed below.

3.1.3. On the effect of a truncated modal space

Some issues in dealing with a truncated modal space are now addressed.

Monitoring with a few modes. Many practical situations correspond to the case where actual data are generated by a system of high order. Controlling the computational complexity of the processing of the collected data is a standard monitoring requirement, which includes the reduction from the analytical model to the experimental model. In the same spirit as a truncated modal space is handled when estimating a flexibility matrix from a few of the lower frequencies [1,5], the actual computation of residual (19) is achieved using a reference parameter vector θ_0 containing only a moderate number of modes and a moderate number of components of associated mode shapes.

[1] Technical arguments for the $\sqrt{n}$ factor can be found in Ref. [17].

Model reduction. The model reduction issue can be investigated further along the following lines. When the actual data are generated by a system of higher order than that of the nominal model θ_0, or equivalently when the nominal model has reduced order, a new question arises: what does it mean for a nominal model θ_0, to match a given data sample when model reduction is enforced? Of course, system theoretic characterization (16), or (18), is no longer valid, and the same is true for the definition of the residual in Eq. (19). Other definitions are needed [9], as sketched now.

Since, $rank\ \mathscr{H}_{p+1,q} > rank\ \mathscr{O}_{p+1}(\theta_0) \stackrel{\text{def}}{=} m$, condition (16) for perfect matching cannot be satisfied. What can be required, instead, is that the left kernel space of $\mathscr{O}_{p+1}(\theta_0)$ is orthogonal to the mth order principal subspace[2] of $\mathscr{H}_{p+1,q}$. Therefore, let

$$W_1\, \mathscr{H}_{p+1,q}\, W_2^{\mathrm{T}} = (P_m \quad \overline{P_m})\, D V^{\mathrm{T}}, \tag{21}$$

where P_m collects the first m left singular vectors of $W_1\, \mathscr{H}_{p+1,q}\, W_2^{\mathrm{T}}$. Then, Eq. (18) is replaced by

$$U^{\mathrm{T}}\, S^{\mathrm{T}}(\theta_0) P_m\, D V^{\mathrm{T}} = 0,$$

and the residual is computed as

$$\zeta_n(\theta_0) \stackrel{\text{def}}{=} \sqrt{n}\, vec(U^{\mathrm{T}}\, S^{\mathrm{T}}(\theta_0)\, \hat{P}_m \hat{D} \hat{V}^{\mathrm{T}}) \tag{22}$$

with obvious notations, namely: $W_1\, \hat{\mathscr{H}}_{p+1,q}\, W_2^{\mathrm{T}} = (\hat{P}_m \quad \overline{\hat{P}_m})\, \hat{D} \hat{V}^{\mathrm{T}}$. It turns out that, because of our practical implementation of residual (19), and especially how the integer index p for $\mathscr{H}_{p+1,q}$ is selected, the use of residual (22) does not seem to lead to any significant performance improvement. This suggests that the test based on Eq. (19) is itself somehow robust to model reduction (since the above conceptual attempt to overcome this issue does not bring practical improvement).

3.2. Residual sensitivities and residual uncertainty

A natural approach to analyze the effect of deviations is to compute sensitivities in terms of gradients. This has been advocated for vibration monitoring as well [2]. Since our approach deals with small deviations, computing gradients is especially relevant. Because the approach is statistical, it is natural to consider the mean value (expectation) of those gradients. On the other hand, the sensitivity matrix of the residual with respect to damages should be examined in the light of the variance of (or the uncertainty in) the components of the residual vector. Within a statistical approach, it is also natural to take into account the possible correlations among those components. This is done now. The off-line and on-board computation stages of the proposed method are distinguished, in 3.3 and 3.4, respectively.

[2] The mth order principal subspace of a matrix is the subspace spanned by the left singular vectors associated with its m largest singular values.

3.3. Off-line computations

The sensitivity of the residual with respect to modal changes is computed, and shown to enjoy a useful invariance property with respect to the normalization of the mode shapes. Then the computation of the residual covariance matrix is discussed.

3.3.1. Computing the residual sensitivities to modal changes

The mean sensitivity of residual ζ_n with respect to θ is defined as

$$\mathcal{J}(\theta_0) \stackrel{\text{def}}{=} -1/\sqrt{n}\; \partial/\partial\theta \mathbf{E}_{\theta_0} \zeta_n(\theta)|_{\theta=\theta_0} \tag{23}$$

$$= +1/\sqrt{n}\; \partial/\partial\theta \mathbf{E}_{\theta} \zeta_n(\theta_0)|_{\theta=\theta_0}, \tag{24}$$

where the last equality results from Eq. (20). From Eq. (24) and using Eq. (12), it can be shown that [9]

$$\mathcal{J}(\theta_0) = (W_2 \otimes U^{\mathrm{T}}\, S^{\mathrm{T}}(\theta_0)\, W_1)(\mathcal{H}^{\mathrm{T}}_{p+1,q}\, \mathcal{O}^{\dagger\mathrm{T}}_{p+1}(\theta_0) \otimes I_{(p+1)r})\mathcal{O}'_{p+1}(\theta_0), \tag{25}$$

where $\mathcal{O}^{\dagger}_{p+1}(\theta_0)$ is the pseudo-inverse of $\mathcal{O}_{p+1}(\theta_0)$, and where

$$\mathcal{O}'_{p+1}(\theta_0) \stackrel{\text{def}}{=} \partial/\partial\theta vec \mathcal{O}_{p+1}(\theta_0)$$

$$= \left(\begin{array}{ccc|ccc} \Lambda'^{(p)}_1 \otimes \varphi_1 & & 0 & \Lambda^{(p)}_1 \otimes I_r & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Lambda'^{(p)}_m \otimes \varphi_m & 0 & & \Lambda^{(p)}_m \otimes I_r \end{array}\right) \tag{26}$$

with $\Lambda^{(p)\mathrm{T}}_i \stackrel{\text{def}}{=} (1\ \lambda_i\ \lambda_i^2\ \ldots\ \lambda_i^p)$, $\Lambda'^{(p)\mathrm{T}}_i \stackrel{\text{def}}{=} (0\ 1\ 2\,\lambda_i\ \ldots\ p\,\lambda_i^{p-1})$ for $1 \leqslant i \leqslant m$.

A consistent estimate $\hat{\mathcal{J}}$, based on a data sample, results from substituting $\hat{\mathcal{H}}$ for $\mathcal{H}$ in Eq. (25):

$$\hat{\mathcal{J}}(\theta_0) = (W_2 \otimes U^{\mathrm{T}}\, S^{\mathrm{T}}(\theta_0)\, W_1)(\hat{\mathcal{H}}^{\mathrm{T}}_{p+1,q}\, \mathcal{O}^{\dagger\mathrm{T}}_{p+1}(\theta_0) \otimes I_{(p+1)r})\mathcal{O}'_{p+1}(\theta_0). \tag{27}$$

Note that all the terms in Eq. (27) should be computed when the reference parameter θ_0 is identified, and using the same reference data. Note also that matrix $\mathcal{O}'_{p+1}(\theta_0)$ is full rank $(r+1)m$, as can be checked from Eq. (26). Finally, because the modes and mode shapes are pairwise complex conjugate, the actual implementation of the computation above should take advantage of the real and imaginary parts, as made explicit in the appendix.

3.3.2. The sensitivity is invariant with respect to mode-shapes normalization

The sensitivity matrix $\mathcal{J}(\theta_0)$ and its estimate $\hat{\mathcal{J}}(\theta_0)$ enjoy a practically useful invariance property: they do *not* depend on the particular normalization of the eigenvectors φ_λ stacked in θ_0 defined in Eq. (10). Actually, multiplying the φ_λ's by constant complex numbers amounts to post-multiply observability matrix $\mathcal{O}_{p+1}(\theta_0)$ in Eq. (15) by an invertible diagonal matrix D, to post-multiply matrix $\mathcal{O}^{\dagger\mathrm{T}}_{p+1}(\theta_0)$ by D^{-1}, and to pre-multiply matrix $\mathcal{O}'_{p+1}(\theta_0)$ by $(D \otimes I_{(p+1)r})$. And all the terms in D cancel out in Eq. (27). It should be noted that this invariance property only holds true for damage detection. For damage localization, some care should be taken of mode shapes normalization, as explained in Section 5.2.

3.3.3. Computing the residual uncertainty

The residual covariance matrix is: $\Sigma(\theta_0) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mathbf{E}_{\theta_0}(\zeta_n\,\zeta_n^{\mathrm{T}})$, where it is assumed that the limit exists. Matrix Σ captures the uncertainty in ζ_n due to estimation errors. Actually, the covariance matrix of the error in estimating θ_0 can be shown to be this $\Sigma(\theta_0)$ as well [26].

The estimation of covariance matrix Σ is somewhat tricky [27,28]. The following estimate is used in the experiments below. Assuming the whole sample size as $N \approx K\ell$, the data sample is partitioned into K segments with size ℓ, and the following estimate is computed:

$$\hat{\Sigma} = \frac{1}{K\ell}\sum_{k=1}^{K} \zeta_k\,\zeta_k^{\mathrm{T}}, \tag{28}$$

where ζ_k is residual (19) computed on segment k, using the data sample $Y_{(k-1)\ell+1}, \ldots, Y_{k\ell}$.

It should be noted that this estimate contains the excitation, and thus is affected by changes in the excitation. Therefore, and this is confirmed in the asymptotic Gaussianity theorem below, it is preferable to estimate it after collecting a new data sample. However, for the sake of reducing the computational complexity, it is often estimated prior to testing, using data on the safe system. The drawback of the latter approach over the former is that the χ^2-test below may detect changes due to the excitation and not to the structural properties.

3.4. On-board residual analysis

Testing if $\theta = \theta_0$ holds true requires the knowledge of the probability distribution of $\zeta_n(\theta_0)$. Unfortunately, this distribution is generally unknown. One way to circumvent this difficulty is to assume close hypotheses

$$\text{(Safe) } \mathbf{H}_0 : \theta = \theta_0 \text{ and (Damaged) } \mathbf{H}_1 : \theta = \theta_0 + \delta\theta/\sqrt{n}, \tag{29}$$

where vector $\delta\theta$ is unknown, but fixed. Note that for large n, hypothesis $\mathbf{H}_1$ corresponds to *small* deviations in θ. This is known under the name of statistical local approach, of which the main result is the following [7,26,27].

3.4.1. The residual is Gaussian

Provided that $\Sigma(\theta_0)$ is positive definite, the residual ζ_n in Eq. (19) is asymptotically Gaussian distributed with the same covariance matrix $\Sigma(\theta_0)$ under both $\mathbf{H}_0$ and $\mathbf{H}_1$; that is [9]

$$\zeta_n(\theta_0) \xrightarrow[n\to\infty]{} \begin{cases} \mathcal{N}(0,\, \Sigma(\theta_0)) & \text{under } \mathbf{H}_0, \\ \mathcal{N}(\mathcal{J}(\theta_0)\,\delta\theta,\, \Sigma(\theta_0)) & \text{under } \mathbf{H}_1. \end{cases} \tag{30}$$

As seen in Eq. (30), a deviation $\delta\theta$ in the system parameter θ is reflected into a change in the mean value of residual ζ_n, which switches from zero (in the undamaged case) to $\mathcal{J}(\theta_0)\delta\theta$, as expected, in case of small damage. Note that matrices $\mathcal{J}(\theta_0)$ and $\Sigma(\theta_0)$ depend on neither the sample size n nor the fault vector $\delta\theta$ in hypothesis $\mathbf{H}_1$. Thus, it is not needed to re-estimate them when testing the hypotheses after collecting a new data sample, they can be estimated prior to testing, using data on the safe system (exactly as the reference parameter θ_0). In case of *non-stationary* excitation, a similar result has been proven, for scalar output signals, and with matrix Σ estimated on newly collected data [29].

3.4.2. On-board χ^2-test for damage detection

Let $\hat{\mathscr{J}}$ and $\hat{\Sigma}$ be consistent estimates of $\mathscr{J}(\theta_0)$ and $\Sigma(\theta_0)$, and assume additionally that $\mathscr{J}(\theta_0)$ is full column rank (f.c.r.). Then, thanks to Eq. (30), deciding that residual ζ_n is *significantly* different from zero, stated as testing between the hypotheses $\mathbf{H}_0$ and $\mathbf{H}_1$ in Eq. (29), can be achieved with the aid of the following χ^2-test statistics:

$$\chi_n^2 \stackrel{\text{def}}{=} \zeta_n^{\mathrm{T}}\, \hat{\Sigma}^{-1}\, \hat{\mathscr{J}}(\hat{\mathscr{J}}^{\mathrm{T}}\, \hat{\Sigma}^{-1}\, \hat{\mathscr{J}})^{-1} \hat{\mathscr{J}}^{\mathrm{T}}\, \hat{\Sigma}^{-1}\, \zeta_n \tag{31}$$

which should be compared to a threshold. In Eq. (31), the dependence on θ_0 has been removed for simplicity. The only term which should be computed after data collection is the residual ζ_n in Eq. (19). Test statistics χ_n^2 is asymptotically distributed as a χ^2-variable, with *rank*($\mathscr{J}$) degrees-of-freedom. From this, a threshold for χ_n^2 can be deduced, for a given false alarm probability. The non-centrality parameter of this χ^2-variable under $\mathbf{H}_1$ is $\delta\theta^{\mathrm{T}}\mathscr{J}^{\mathrm{T}}\Sigma^{-1}\mathscr{J}\delta\theta$. This provides us with the theoretical mean value of χ_n^2 when a damage is present: number of degrees-of-freedom plus non-centrality parameter. In practice, however, the actual values of χ_n^2 are much higher than those theoretical values. How to select a threshold for χ_n^2 from histograms of empirical values obtained on data for undamaged cases is explained in Ref. [12].

3.4.3. The χ^2-test is invariant with respect to design matrices

The test in Eq. (31) enjoys a nice invariance property with respect to the choice of the design matrices in the subspace-based approach. The three design matrices U, W_1, W_2 are made explicit in the notation $\zeta_{n;U,W_1,W_2}(\theta_0)$ for residual (19). A straightforward calculation shows that

$$\zeta_{n;U,W_1,W_2} = (W_2 \otimes V_1)\zeta_{n;I,I,I}, \tag{32}$$

where $\otimes$ is the Kronecker product, and V_1 is any invertible matrix such that: $U^{\mathrm{T}}S^{\mathrm{T}}(\theta_0)W_1 = V_1 S^{\mathrm{T}}(\theta_0)$; for example, $V_1 = U^{\mathrm{T}}S^{\mathrm{T}}(\theta_0)W_1 S(\theta_0)$ [30]. Now, if $\tilde{\zeta} \stackrel{\text{def}}{=} A\zeta$, then $\tilde{\mathscr{J}} = A\mathscr{J}$ and $\tilde{\Sigma} = A\Sigma A^{\mathrm{T}}$, as can easily be checked, and using obvious notations. Therefore, if $\chi^2_{n;U,W_1,W_2}$ denotes the χ^2-test (31) associated with $\zeta_{n;U,W_1,W_2}$, from Eq. (32) it comes that

$$\chi^2_{n;U,W_1,W_2} = \chi^2_{n;I,I,I}$$

since the invertible matrix $(W_2 \otimes V_1)$ factors out in Eq. (31). Thus, for damage detection, *all the proposed subspace-based methods are equivalent*, when using the true system order.

4. Modal diagnosis

Modal diagnosis consists in determining which eigenfrequencies and associated mode shapes have been affected by the damage. This problem may be addressed as an estimation problem, based on modal identification in the pre- and post-damage stages. Typically, the changes in the frequencies obtained from experimental data are then compared with the sensitivity of modal parameters obtained from an analytical model [1,2]. In the proposed approach, modal diagnosis is stated as a detection problem instead. Moreover, no analytical model is handled at this stage. The rationale is the same as for addressing the damage detection problem above: having a (usually identified) reference modal parameter θ_0 in one hand and a new data sample in the other, decide which modes and mode shapes have deviated from their reference values.

4.1. Residual sensitivities and residual uncertainty

As for the damage detection problem, in a statistical framework the key issue is again to assess the significance of the residual sensitivity to specified modal changes with respect to the residual uncertainty. In other words, directional tests, focussed in specific directions of the modal space, should be designed. At this point, it should be noted that the sensitivity of the residual ζ_n with respect to a specified mode and associated mode shape can be extracted as the corresponding columns of the Jacobian matrix $\mathscr{J}(\theta_0)$ of which an estimate is given in Eq. (27).

4.2. On-board χ^2-test for modal diagnosis

Thus let $\hat{\mathscr{J}}_j$ be the estimated Jacobian matrix corresponding to mode and mode shape j. This is obtained by picking up the corresponding columns in matrix $\mathscr{O}'_{p+1}(\theta_0)$ in Eq. (26). An alternative, and equivalent, computation is detailed in the appendix, which leads to Eq. (48). The counterpart of test (31), namely the directional or sensitivity test focussed on this mode j, writes

$$\chi_n^{(j)2} \stackrel{\text{def}}{=} \zeta_n^{\mathrm{T}} \hat{\Sigma}^{-1} \hat{\mathscr{J}}_j (\hat{\mathscr{J}}_j^{\mathrm{T}} \hat{\Sigma}^{-1} \hat{\mathscr{J}}_j)^{-1} \hat{\mathscr{J}}_j^{\mathrm{T}} \hat{\Sigma}^{-1} \zeta_n. \tag{33}$$

As for damage detection above, the only term which should be computed after data collection is the residual ζ_n defined in Eq. (19). Of course, it is often not easy to interpret the results provided by a collection of such tests focussed on specific modal parameters. Those tests are of interest only when some modal parameters are subject to changes much larger than the others. This is the case for flutter monitoring [31].

5. Damage localization

Damage localization consists of determining which part of the structure has been affected by the damage, more precisely which (groups of) elements of the structural parameters matrices (e.g., M, C, K) have changed. This problem is often addressed as an inverse estimation problem, based on an analytical model in the pre-damage stage and on modal identification in the post-damage stage. Typically, the deviations in the structural parameters corresponding to the observed deviation in the modal parameters are searched for using model updating techniques [32–34]. In the proposed approach, damage localization is stated as a detection problem instead. Of course, an analytical model is handled at this stage. The rationale is similar to the approach for modal diagnosis above: having a (usually identified) reference modal parameter θ_0 and a reference structural (FEM) analytical model in one hand, and a new data sample in the other, decide which structural parameters have deviated from their reference values.

5.1. Residual sensitivities and residual uncertainty

As for the damage detection and modal diagnosis problems, in a statistical framework the key issue is again to assess the significance of the residual sensitivity to specified structural changes with respect to the residual uncertainty. As made explicit in Eq. (30), the mean value of residual ζ_n under the hypothesis $\mathbf{H}_1$ of a small deviation $\delta\theta$ in the parameter θ from a reference value θ_0, is

$$\mathbf{E}_1(\zeta_n) = \mathcal{J}(\theta_0)\delta\theta. \tag{34}$$

Under the assumption of small deviation again, the relation holds:

$$\delta\theta \approx \mathcal{J}_{\theta\Psi}\delta\Psi, \tag{35}$$

where Ψ is the vector of (FEM) structural parameters to be monitored, and $\mathcal{J}_{\theta\Psi}$ is the Jacobian matrix containing the sensitivities of the modes and mode shapes with respect to those parameters.

In other words, the damage localization problem is addressed by plugging aggregated sensitivities of the modes and mode shapes with respect to (FEM) structural parameters in the setting used for damage detection. Thus, plugging Eq. (35) into Eq. (34), it can be tested, with the aid of a χ^2-test again, whether the deviation in the residual ζ_n: $\mathbf{E}_1(\zeta_n) = \mathcal{J}(\Psi)\delta\Psi$, where

$$\mathcal{J}(\Psi) \stackrel{\text{def}}{=} \mathcal{J}(\theta_0)\mathcal{J}_{\theta\Psi} \tag{36}$$

is significant or not. This results in directional tests, focussed in specific directions of the structural space, which perform the same type of damage-to-noise sensitivity analysis of the residual as for damage detection and modal diagnosis. These tests deliver damage diagnostics and localization information, without solving any inverse problem for model updating.

Of course, since the dimension of the structural parameter space is much higher than the dimension of the modal parameter space, the $\delta\theta$ in Eq. (35) are linearly dependent, even if the $\delta\psi$'s are not. The idea [13,35,36] is somehow to cluster the deviations $\delta\psi$ in the structural parameter space. The steps of this damage localization approach are now described in detail. The off-line and on-board computation stages are distinguished, in 5.2 and 5.3, respectively.

5.2. Off-line computations

The off-line stage is devoted to the computation of the residual sensitivities with respect to structural changes. First the different parameterizations and Jacobians which are needed for this purpose are explained. Then three design steps are described: computing sensitivities (35), matching theoretical and actual sensitivities, aggregating the sensitivities.

5.2.1. Computing the residual sensitivities to structural changes

For computing the residual sensitivity with respect to structural changes given in Eq. (36), the Jacobian $\mathcal{J}_{\theta\Psi}$ defined in Eq. (35) must be computed. For this purpose, in addition to the structural parameterization Ψ, two other parameterizations are needed: $\Phi_i^{(d)} \stackrel{\text{def}}{=} \theta$, the set of the (discrete) *identified* modal parameters, and Φ_a, the set of the (continuous) *analytical* modal parameters computed from Eq. (2). Since the former is a discrete time parameterization and the latter a continuous time parameterization, the continuous time counterpart Φ_i of $\Phi_i^{(d)}$, obtained using Eq. (8), is also needed. Moreover, the re-parameterization of the modes in terms of frequencies and damping coefficients are needed for both Φ_i and Φ_a, which are noted ϑ_a and ϑ_i, respectively.

It should be stressed that, when the system is assumed to be conservative, namely $C = 0$, which is often the case in FEM models, Φ_a contains all the frequencies, but *not* the damping coefficients, of the structure, and all the mode shapes, which are real and usually mass normalized. Also, the

FEM mode shapes have as many components as the *total* number of available sensors. Whereas ϑ_i contains those of the frequencies and associated mode shapes contained in the signature θ_0 which turn out to match with modes in ϑ_a. Also, the mode shapes have as many components as the *actual* number of sensors used and are *not* mass normalized.

Altogether, the sensitivity $\mathcal{J}(\Psi)$ defined in Eq. (36) writes

$$\mathcal{J}(\Psi) = \mathcal{J}(\theta_0)\mathcal{J}_{\Phi_i^{(d)}\Phi_i}\mathcal{J}_{\Phi_i\vartheta_i}\mathcal{J}_{\vartheta_i\vartheta_a}\mathcal{J}_{\vartheta_a\Phi_a}\mathcal{J}_{\Phi_a\Psi}, \tag{37}$$

where $\mathcal{J}_{\Phi_i^{(d)}\Phi_i}$ is the Jacobian of transformation (8) of the discrete modes into continuous ones; $\mathcal{J}_{\Phi\vartheta}$ is the Jacobian of conversion (3) of the complex modes into continuous frequencies and damping coefficients, and $\mathcal{J}_{\vartheta\Phi}$ is the Jacobian of the inverse conversion; $\mathcal{J}_{\vartheta_i\vartheta_a}$ corresponds to the manual matching between the identified modes and the analytical (computed) ones; $\mathcal{J}_{\Phi_a\Psi}$ represents the sensitivities of analytical modes to changes in structural parameters. The three former Jacobians are computed in the appendix, whereas the two latter are derived in this section.

At this point, one comment is in order about the damage detection χ^2-test in Eq. (31) and the modal diagnosis test in Eq. (33). These tests are computed using $\hat{\mathcal{J}}\mathcal{J}_{\Phi_i^{(d)}\Phi_i}\mathcal{J}_{\Phi_i\vartheta_i}$ instead of $\hat{\mathcal{J}}$. But this does not make any difference, since the two matrices $\mathcal{J}_{\Phi_i^{(d)}\Phi_i}$ and $\mathcal{J}_{\Phi_i\vartheta_i}$ are square invertible and cancel out in Eqs. (31) and (33).

5.2.2. *Computing the sensitivities of analytical modes to structural changes*

The derivation of $\mathcal{J}_{\Phi_a\Psi}$ is described now. Differentiating the second equation in Eq. (2), it comes:

$$\delta\mu(2\mu M + C)\phi + (\mu^2 M + \mu C + K)\delta\phi + \mu^2\delta M + \mu\delta C + \delta K = 0 \tag{38}$$

[35–38]. When the matrices M, C, K are symmetric, pre-multiplying Eq. (38) with ϕ^{T}, and using $\phi^{\mathrm{T}}(\mu^2 M + \mu C + K) = 0$ which results from that symmetry, yield

$$\delta\mu = -\frac{\phi^{\mathrm{T}}(\mu^2\delta M + \mu\delta C + \delta K)\phi}{\phi^{\mathrm{T}}(2\mu M + C)\phi}. \tag{39}$$

In case of an asymmetric system, the results of Ref. [39] should be used instead. They generalize the approach of Ref. [40], which is based on a complete modal basis. An overview on different types of approximation methods for handling incomplete modal bases can be found in Ref. [41].

Plugging Eq. (39) into Eq. (38) yields the following linear system in $\delta\phi$:

$$(\mu^2 M + \mu C + K)\delta\phi = -\delta\mu(2\mu M + C)\phi - (\mu^2\delta M + \mu\delta C + \delta K)\phi. \tag{40}$$

This system has no unique solution: if $\delta\phi$ is a solution, then $\delta\phi + \alpha\phi$, where α is a real constant, is also a solution. The solution $\delta\phi$ that is orthogonal to ϕ is selected, namely

$$\phi^{\mathrm{T}}\delta\phi = 0 \tag{41}$$

and then pre-multiplied by the observation matrix

$$L\delta\phi \tag{42}$$

since the mode shapes are of interest. For each δM, δC and δK, Eqs. (39)–(42) yield the corresponding change in the whole modal parameters set Φ_a. This leads to the first order sensitivity relationship: $\delta\Phi_a = \mathcal{J}_{\Phi_a\Psi}\delta\Psi$, where each column of sensitivity matrix $\mathcal{J}_{\Phi_a\Psi}$ corresponds to a change in a structural parameter.

At this point, a comment is in order, on mode-shape normalization. As is obvious from Eq. (40), the actual value of $\delta\phi$ depends on the normalization chosen for ϕ. Therefore, it is important here to work with analytical mode shapes normalized in the same way as the identified ones. Typically, the first component is fixed equal to one.

5.2.3. Matching theoretical and actual sensitivities

How to match the changes in ϑ_a with the changes in ϑ_i is explained now. It is well known that there is usually a discrepancy between the numerical modes ϑ_a provided by the analytical (FEM) model, and the identified modes ϑ_i. Moreover, whatever the modal identification method is, there is an estimation error on ϑ_i. Consequently, $\vartheta_i \neq \vartheta_a$ in general. However, this discrepancy between ϑ_i and ϑ_a is not crucial within our damage localization approach. Actually, ϑ_i is the reference modal signature and must be accurately determined, which is ensured by the subspace-based identification method, whereas ϑ_a is only used for computing a Jacobian matrix of sensitivities (change directions). A small error in such a direction enters the algorithm at a second order level in the residual ζ_n. It is expected that this does not corrupt the localization delivered by the method.

Moreover, it should be noticed that the matching between ϑ_i and ϑ_a is generally *partial*. On one hand, only the first few natural modes are excited and/or observed, whereas p degrees of freedom finite element model yields p modes, with p often very large (several hundreds). Consequently, ϑ_i is at best strictly included in ϑ_a. On the other hand, ϑ_i may contain modes which are not related to the eigen parameters of the structure. These modes appear in the presence of an harmonic excitation (as it is the case for rotating machineries for example Ref. [6]), or when the effect of the environment should be considered as an unknown coloured noise, instead of a white noise as in Eq. (1).

Therefore, it is assumed that: $\delta\vartheta_i = \mathscr{J}_{\vartheta_i\vartheta_a}\delta\vartheta_a$, where $\mathscr{J}_{\vartheta_i\vartheta_a}$ is a selection matrix which performs the correlation between the analytical (computed) modes and the identified ones.

5.2.4. Aggregating the sensitivities

The (M, C, K) parameterization has generally many more parameters than the modal model. Thus, there are many more columns than lines in matrix $\mathscr{J}$ in Eq. (37). Moreover, using a small number of sensors, it is not reasonable to expect the discrimination of all possible structural causes of a given deviation detected by the global damage detection test (31). To circumvent this difficulty, the idea is to aggregate the columns of $\mathscr{J}$ into clusters, which play the role of macro-failures, and for each cluster to define a barycentre, which plays the role of a Jacobian to be plugged in Eq. (31).

The χ^2-metric: In order to make the aggregation operation coherent with the χ^2 decision stage, the metric chosen for performing the clustering is the metric of the χ^2-test. More precisely, let the jth *change direction* be defined as the vector

$$\mathscr{J}_j = \hat{\Sigma}_n^{-\mathrm{T}/2}\mathscr{J}(\theta_0)\mathscr{J}_{\Phi_i^{(d)}\Phi_i}\mathscr{J}_{\Phi_i\vartheta_i}\mathscr{J}_{\vartheta_i\vartheta_a}\mathscr{J}_{\vartheta_a\Phi_a}\mathscr{J}_{\Phi_a\Psi}(j), \tag{43}$$

where $\mathscr{J}_{\Phi_a\Psi}(j)$ is the jth column of $\mathscr{J}_{\Phi_a\Psi}$, and where $\hat{\Sigma}_n^{-\mathrm{T}/2}$ is the 'square root' of the inverse of the covariance matrix $\hat{\Sigma}_n$: $\hat{\Sigma}_n^{-1} = \hat{\Sigma}_n^{-1/2}\hat{\Sigma}_n^{-\mathrm{T}/2}$. Note that such a decomposition is always possible since $\hat{\Sigma}_n$ is guaranteed to be strictly positive definite from its numerical computation in (28).

Furthermore, the norm and the scalar product of the $\mathcal{J}_j$'s are defined as

$$\|\mathcal{J}_j\|^2 = \mathcal{J}_j^{\mathrm{T}} \mathcal{J}_j, \quad d_{ij} = \frac{\mathcal{J}_i^{\mathrm{T}} \mathcal{J}_j}{\|\mathcal{J}_i\| \|\mathcal{J}_j\|}. \tag{44}$$

Removing the small vectors: Vectors with a very low magnitude are likely to blur the results of the aggregation. Consequently, prior to clustering, small vectors are rejected using the following rule, based on the contrast (ratio) between the expectations of the directional test under no damage and small damage assumptions [13]. By definition, vector $\mathcal{J}_j$ corresponds to a change with rate 1 in physical parameter j. Therefore, up to a first order approximation, for a change with magnitude ϱ in the jth direction, the change vector is $\varrho\mathcal{J}_j$. The expectation of the corresponding sensitivity test (33) is equal to 1 under no damage hypothesis and to $(1 + \varrho^2 \mathcal{J}_j^{\mathrm{T}} \mathcal{J}_j)$ under this damage hypothesis. The above mentioned contrast is thus: $1 + \varrho^2 \mathcal{J}_j^{\mathrm{T}} \mathcal{J}_j$. Consequently, considering that a damage with magnitude ϱ in direction $\mathcal{J}_j$ is detectable provided that this contrast exceeds a threshold ϵ_1, the minimum magnitude of a damage for being detectable should be: $\varrho_{min} = \sqrt{\epsilon_1 - 1}/\|\mathcal{J}_j\|$. A damage vector $\mathcal{J}_j$ will be rejected if this minimum damage magnitude cannot be reached, e.g., because greater than a percentage of variation on the physical parameters, namely if: $\varrho_{min} \geqslant \epsilon_2/100$. Altogether, the rejection rule for small vectors writes: $\|\mathcal{J}_j\| \leqslant 100\sqrt{\epsilon_1 - 1}/\epsilon_2$, where ϵ_1, ϵ_2 are two thresholds selected by the user.

Clustering the remaining vectors: Since change directions are of interest, rather than change magnitudes, the change vectors to be clustered are normalized within this metric. Therefore, the aggregation process should work on the unit sphere, and a classification method able to handle this geometry is needed. For this reason, a vector quantization method [42,43] of common use in speech processing has been chosen [13]. This method performs a hierarchical classification, while controlling the variability within the classes. For each class, a barycentre C_j is computed. This aggregation mechanism can thus be thought of as a statistical sub-structuring.

5.3. On-board χ^2-test for damage localization

For performing damage localization, it should be assessed, for each barycentre C_j, whether the corresponding damage is significant or not. This problem is similar to the damage detection and modal diagnosis problems addressed in Sections 3.4 and 4.2, respectively, and is solved in the same manner. Because of Eq. (43), the following normalized residual should be considered: $\tilde{\zeta}_n \overset{\text{def}}{=} \hat{\Sigma}_n^{-\mathrm{T}/2} \zeta_n$. Then the counterpart of the χ^2-test in Eq. (33) is easily shown to write

$$\chi_n^2(j) = \tilde{\zeta}_n^{\mathrm{T}} \frac{C_j C_j^{\mathrm{T}}}{\|C_j\|^2} \tilde{\zeta}_n.$$

Its number of degrees of freedom is equal to $rank(\mathcal{J}_j)$.

Assume that $\chi_n^2(j)$ exceeds a given threshold. Then, all the structural elements within the class corresponding to the barycentre C_j are possible causes of the detected damage.

6. Discussion and relation to other works

Several issues are addressed in this section, concerning the design and the handling of residuals for damage detection and localization. First some further comments on the design of residuals for fault or damage detection are provided. Then the subspace-based damage detection residual introduced in Section 3 is related to some other residuals, used for damage localization or model updating. Finally, some lessons learnt from practical experiments are provided.

Residuals for fault detection: Model-based approaches to fault detection and isolation have been investigated, in Refs. [44–46] to mention but a few. They build on discrepancies between process model outputs and measured outputs, generically called *residuals*, often generated as an output prediction error:

$$\varepsilon_k(\theta) \stackrel{\text{def}}{=} Y_k - \hat{Y}_{k|k-1}(\theta), \tag{45}$$

where $\hat{Y}_{k|k-1}(\theta)$ is a one-step ahead prediction of the output data, computed on the basis of the parameterized model. The residual ε_k is then evaluated through a comparison with given thresholds. From a conceptual point of view, however, this type of residual suffers from the following limitation. If the system is linear, written in an input–output or state-space form, and whatever the estimation method is, the prediction $\hat{Y}$ is a *linear* combination of measured inputs and outputs. Stated otherwise, residual (45) is a *first order* statistics in that case. But, from statistical inference theory, it is known that for performing inference about second-order characteristics—here, (modal) vibrating characteristics, or equivalently the eigenstructure of a linear state space system—, it is mandatory to use (at least) second order statistics, that is covariances. Using linear combinations of the output data is *not* sufficient (in the statistical sense). This might be contrasted with some of the symptoms discussed in Ref. [47]. It should be noted that the subspace-based residual defined in Eq. (19) is actually a linear combination of the output *covariances*, and indeed is aimed at monitoring the system eigenstructure, as desired.

The above remark does not mean, however, that the prediction error (45) should not play any role in residual generation. As clearly stated in the system identification literature [48–50], a parameter estimate should be updated with the aid of the *gradient* of the squared prediction error with respect to the parameter: $-1/2\partial(\varepsilon_k^{\mathrm{T}}(\theta)\,\varepsilon_k(\theta))/\partial\theta$. And if the faults or damages affect the dynamics of the system, a residual built on that gradient is relevant too.

Subspace-based residuals and other health monitoring residuals: How the proposed subspace-based residuals relate to some other damage localization and model updating residuals in the literature is now discussed. In Ref. [1], damage detection is based on changes in the flexibility matrix computed as $\mathscr{F} = \Phi\Omega^{-1}\Phi^{\mathrm{T}}$, where diagonal matrix Ω, called modal stiffness matrix in the case of proportional damping, is the stiffness matrix of the single degree-of-freedom system resulting from decoupling of the equations by transformation to modal co-ordinates. The damage locating vectors introduced in Ref. [5] are the *last right* singular vectors, namely the singular vectors associated with singular value 0, of the change $\delta\mathscr{F} \stackrel{\text{def}}{=} \mathscr{F}_i^1 - \mathscr{F}_i^0$ in the measured flexibility matrix computed as $\mathscr{F} \stackrel{\text{def}}{=} K^{-1}$. In other words, these vectors form the kernel space of matrix $\Delta\mathscr{F}$. When viewed as loads on the system, they lead to stress fields identically zero over the damaged elements. This property is considered in Ref. [5] as a damage localization ability. An important limitation is that this is basically an input–output approach. However, the method can be extended to flexibility proportional matrices which can be computed from output-only data [51]. On the other hand, the subspace-based residuals ζ in Eq. (19) are derived from the *first left* singular vectors of the observability matrix in modal basis $\mathcal{O}(\theta_0)$, from which matrix $S(\theta_0)$ in Eq. (19) is computed. In other words, for computing ζ, the kernel space of matrix $\mathcal{O}^T(\theta_0)$ is needed.

One class of residuals used for model updating in Ref. [32] has the form: $W(\theta_i - \theta_a - \mathcal{J}_{\Phi_a \Psi} \delta \Psi)$, where W is a weighting matrix (design parameters), and subscripts i, a have the same meaning as in 5.2.1. In other words, these residuals are based on discrepancies between identified and analytical modal parameters. This has to be contrasted with the residuals in Eq. (19) which handle altogether the identified modal parameters in θ_0 and newly collected data in $\hat{\mathcal{H}}$. These residuals do not involve any analytical model, and do not require any re-identification of the modal parameters on the new data. Moreover the χ^2-test in Eq. (31) involves the precision in the estimated modal parameters [26]. These test statistics allow assessment of whether a deviation in the reference modal parameter is significant with respect to the inherent inaccuracy in the modal parameter estimate.

In Ref. [33], strong emphasis is put on the physical meaning of the chosen parameterization. It has been argued above, however, that balancing this property with requirements on invariance with respect to changes in the state-space basis on the one hand, and on modal identifiability and detectability properties on the other one, provides other useful points of view.

Another overview of residuals used within model updating methods can be found in Ref. [52].

On the use and physical interpretation of the SVD: It is well known that one major by-product of the singular value decomposition (SVD) is to provide us with various subspaces of interest, among which are the right and left kernels (null space) of the considered matrix [23]. As highlighted in Ref. [47], the use of the SVD for the purpose of damage detection and localization is thus widespread. Moreover, the physical interpretation of the corresponding vectors is an important issue. As mentioned above, the SVD is used in Ref. [5] for extracting the null space of the change $\delta\mathcal{F}$ in the flexibility matrix measured in both undamaged and damaged situations. The singular vectors are interpreted in terms of loads with null stress, and each singular value is interpreted as the difference in the external work done by the associated singular vector. In the approach of this paper, the SVD is also used for extracting the null space of a matrix, namely the transpose of the observability matrix. Moreover, as explained in 3.1.3 and confirmed by extensive numerical experiments, the SVD helps in overcoming non-stationary ambient excitation.

On the mode shapes normalization: The last comment on the design and computation of residuals for health monitoring concerns the mode-shape normalization [41]. As outlined in 3.3.2, the proposed damage detection χ^2-test is invariant with respect to mode-shape normalization, which is a useful property in practice. This has to be contrasted with the approach in Ref. [5].

Structural aggregation: Now some comments are in order on the statistical clustering of the sensitivities of the modes and mode shapes with respect to FEM parameters described in Section 5. It should first be noted, from Eq. (43), that *noise-normalized sensitivities* are handled in the proposed approach. Consequently, the norm and scalar product defined in Eq. (44) introduce a noise-normalized metric for assessing the size of the modes and mode-shapes sensitivities to changes in structural parameters. Here *noise* refers to both measurement noise and modal estimation errors, as pointed out in 3.3.3. This has to be contrasted with the *modal appropriation criterion* (MAC) value, which is, in some sense, an absolute value: the MAC value does not take into account neither any measurement noise, nor any uncertainties in the modes and mode-shape estimates (measured structural behaviour).

Second, it should be stressed that the statistical procedure described in 5.2.4 performs the aggregation of physical parameters, and thus generates what can be considered, in some sense, as super-elements. This procedure might be called statistical substructuring.[3]

Constraints in and lessons from practice: The algorithms have been successfully applied to a wide range of real cases, under non-stationary excitation. From those experiments, it appears that the algorithms have to be tuned during a safe period, from which the covariance and Jacobian matrices, the empirical mean value and dispersion of the test under the no-damage hypothesis are estimated. Theoretically, this period should be as long as possible, for example a few months for the Z24 bridge [12]. But in most cases, where such a long period is not feasible, the test has reacted almost as well [10,11,31,53]. For damage localization, a reasonable FE model should be used, and relevant sensors and modal parameters should be selected [54]: localization is negatively affected when using modal parameters which are not affected by the damage.

7. Conclusion

The damage localization problem has been addressed and stated as a detection problem and not as an inverse estimation problem. The proposed damage localization method is based on both a subspace residual and on a statistical analysis of aggregated sensitivities of the residual to the damages. The key components of the statistical damage detection and damage localization steps have been described in detail. The computations which can be performed off-line at a design stage and the computations which have to be done on-line while processing newly recorded data have been distinguished. The proposed approach has been related to several methods available in the literature. The present method does *not* directly perform the discrimination between non-damaging structural changes such as environmental effects and actual damages. This is known to be a crucial issue in practice [55]. Some preliminary, indirect and empirical attempts for handling and overcoming such effects have been reported in Ref. [12] for the Z24 bridge example. Investigation towards an intrinsic handling and rejection of temperature effects is the topic of ongoing research.

Acknowledgements

The authors are indebted to the editor and the referees, whose careful reading and comments helped improving an earlier version of the paper, and to Albert Benveniste for useful comments. Part of this work has been carried out within the framework of the Eureka projects no 1562 SINOPSYS (Model based **S**tructural monitoring using **in-op**eration **sys**tem identification) coordinated by LMS, Leuven, Belgium, and no 2419 FLITE (**Fli**ght **T**est **E**asy), coordinated by Sopemea, Velizy-Villacoublay, France. The covariance-driven subspace identification algorithm

[3] Substructuring is a procedure that condenses a group of finite elements into one element represented as a matrix (super-element).

has been implemented within two toolboxes: the IN-OP module of the LMS software CADA-X, and the modal analysis module of the free INRIA software Scilab [56–58]. The damage detection and localization methods have been implemented within the modal analysis module of Scilab, and within LMS software environment.

Appendix A. Some Jacobian computations

This appendix is twofold. First the sensitivity $\mathscr{J}(\theta_0)$ to modal changes in Eq. (25), in particular matrix $\mathscr{O}'_{p+1}(\theta_0)$, is explicited. Second, the Jacobians $\mathscr{J}_{\Phi_i^{(d)}\Phi_i}$ of transformation (8) of the discrete modes into the continuous ones, the Jacobian $\mathscr{I}_{\Phi\vartheta}$ of conversion (3) of the complex modes into continuous frequencies and damping coefficients, and the Jacobian $\mathscr{I}_{\vartheta\Phi}$ of the inverse conversion, are computed.

A.1. Computation of $\mathscr{O}'_{p+1}(\theta_0)$ for $\mathscr{J}(\theta_0)$

The modes and mode shapes are pairwise complex conjugate, thus observability matrix (15) writes

$$\mathscr{O}_{p+1}(\theta) = \begin{pmatrix} \Phi & \bar{\Phi} \\ \Phi\Delta & \overline{\Phi\Delta} \\ \vdots & \vdots \\ \Phi\Delta^p & \overline{\Phi\Delta^p} \end{pmatrix}. \tag{A.1}$$

Taking advantage of the real and imaginary parts of the modes and mode shapes, another matrix is introduced, which results from a post-multiplication of $\mathscr{O}_{p+1}(\theta)$ by a permutation matrix:

$$\tilde{\mathscr{O}}_{p+1} \stackrel{\text{def}}{=} \begin{pmatrix} \text{Re}(\Phi) & \text{Im}(\Phi) \\ \text{Re}(\Phi\Delta) & \text{Im}(\Phi\Delta) \\ \vdots & \vdots \\ \text{Re}(\Phi\Delta^p) & \text{Im}(\Phi\Delta^p) \end{pmatrix}.$$

Similarly, from θ defined in Eq. (10), define

$$\tilde{\theta} \stackrel{\text{def}}{=} \begin{pmatrix} \text{Re}(\Lambda) \\ \text{Re}(vec\Phi) \\ \text{Im}(\Lambda) \\ \text{Im}(vec\Phi) \end{pmatrix} = \begin{pmatrix} \text{Re}(\theta) \\ \text{Im}(\theta) \end{pmatrix}.$$

Now let

$$\Theta_{p+1} \stackrel{\text{def}}{=} vec \begin{pmatrix} \Phi \\ \Phi\Delta \\ \vdots \\ \Phi\Delta^p \end{pmatrix}.$$

Then

$$\tilde{\Theta}_{p+1} \stackrel{\text{def}}{=} vec\tilde{\mathcal{O}}_{p+1} = \begin{pmatrix} \mathrm{Re}(\Theta_{p+1}) \\ \mathrm{Im}(\Theta_{p+1}) \end{pmatrix} \tag{A.2}$$

Note that $\tilde{\mathcal{O}}'_{p+1} \stackrel{\text{def}}{=} \partial vec\tilde{\mathcal{O}}_{p+1}/\partial\tilde{\theta} = \partial\tilde{\Theta}_{p+1}/\partial\tilde{\theta}$.

Using Cauchy–Riemann theorem, from Eq. (A.2) it comes that

$$\partial\tilde{\Theta}_{p+1}/\partial\tilde{\theta} = \begin{pmatrix} \partial\mathrm{Re}(\Theta_{p+1})/\partial\mathrm{Re}(\theta) & \partial\mathrm{Re}(\Theta_{p+1})/\partial\mathrm{Im}(\theta) \\ \partial\mathrm{Im}(\Theta_{p+1})/\partial\mathrm{Re}(\theta) & \partial\mathrm{Im}(\Theta_{p+1})/\partial\mathrm{Im}(\theta) \end{pmatrix}.$$

Therefore, the computation of $\mathcal{O}'_{p+1}(\theta_0)$ proceeds as follows: Compute $\mathcal{O}'_{p+1}$ from Eq. (26) using only *half* of the modes and mode shapes; Fill in $M \stackrel{\text{def}}{=} (\mathcal{O}'_{p+1} \;\; i\mathcal{O}'_{p+1})$; Then

$$\mathcal{O}'_{p+1}(\theta_0) = \begin{pmatrix} \mathrm{Re}(M) \\ \mathrm{Im}(M) \end{pmatrix}. \tag{A.3}$$

A.2. Computation of $\mathcal{J}_{\Phi_i^{(d)}\Phi_i}$, $\mathcal{J}_{\Phi\vartheta}$ and $\mathcal{J}_{\vartheta\Phi}$

Since none of transformations (8) and (3) affect the mode shapes, the restriction of the Jacobian to the eigenvectors is, in the three cases, the identity matrix with size $2mr$. Thus, in each case, the focus is first on the restriction of the Jacobians to the eigenvalues.

Computation of $\mathcal{J}_{\Phi_i^{(d)}\Phi_i}$: Using the notation $a = \mathrm{Re}(\mu), b = \mathrm{Im}(\mu), x = \mathrm{Re}(\lambda), y = \mathrm{Im}(\lambda)$, the first equation of Eq. (8) writes: $x = \mathrm{e}^{\tau a}\cos(\tau b), y = \mathrm{e}^{\tau a}\sin(\tau b)$, from which it is deduced that

$$\begin{pmatrix} \partial x/\partial a & \partial x/\partial b \\ \partial y/\partial a & \partial y/\partial b \end{pmatrix} = \tau\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \tau\begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Re}(i\lambda) \\ \mathrm{Im}(\lambda) & \mathrm{Im}(i\lambda) \end{pmatrix} \stackrel{\text{def}}{=} M_\lambda.$$

Then

$$\mathcal{J}_{\Phi_i^{(d)}\Phi_i} = \begin{pmatrix} M_{\lambda_1} & & & \\ & \ddots & & 0 \\ & & M_{\lambda_m} & \\ 0 & & & I_{2mr} \end{pmatrix}.$$

Computation of $\mathcal{J}_{\Phi\vartheta}$: Relation (3) writes equivalently: $a = -2\pi fd/\sqrt{1-d^2}, b = 2\pi f$, from which it is deduced that

$$\begin{pmatrix} \partial a/\partial d & \partial a/\partial f \\ \partial b/\partial d & \partial b/\partial f \end{pmatrix} = \begin{pmatrix} -2\pi f/\sqrt{(1-d^2)^3} & -2\pi d/\sqrt{1-d^2} \\ 0 & 2\pi \end{pmatrix} \stackrel{\text{def}}{=} A.$$

Then

$$\mathscr{I}_{\Phi\vartheta} = \begin{pmatrix} A_1 & & & \\ & \ddots & & 0 \\ & & A_m & \\ 0 & & & I_{2mr} \end{pmatrix}.$$

Computation of $\mathscr{I}_{\vartheta\Phi}$: The Jacobian of relation (3) writes

$$\begin{pmatrix} \partial d/\partial a & \partial d/\partial b \\ \partial f/\partial a & \partial f/\partial b \end{pmatrix} = \begin{pmatrix} -b^2/\sqrt{(a^2+b^2)^3} & -2ab/\sqrt{(a^2+b^2)^3} \\ 0 & 1/2\pi \end{pmatrix} \stackrel{\text{def}}{=} B$$

Then

$$\mathscr{I}_{\vartheta\Phi} = \begin{pmatrix} B_1 & & & \\ & \ddots & & 0 \\ & & B_m & \\ 0 & & & I_{2mr} \end{pmatrix}$$

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